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**Research Paper
RM-307**

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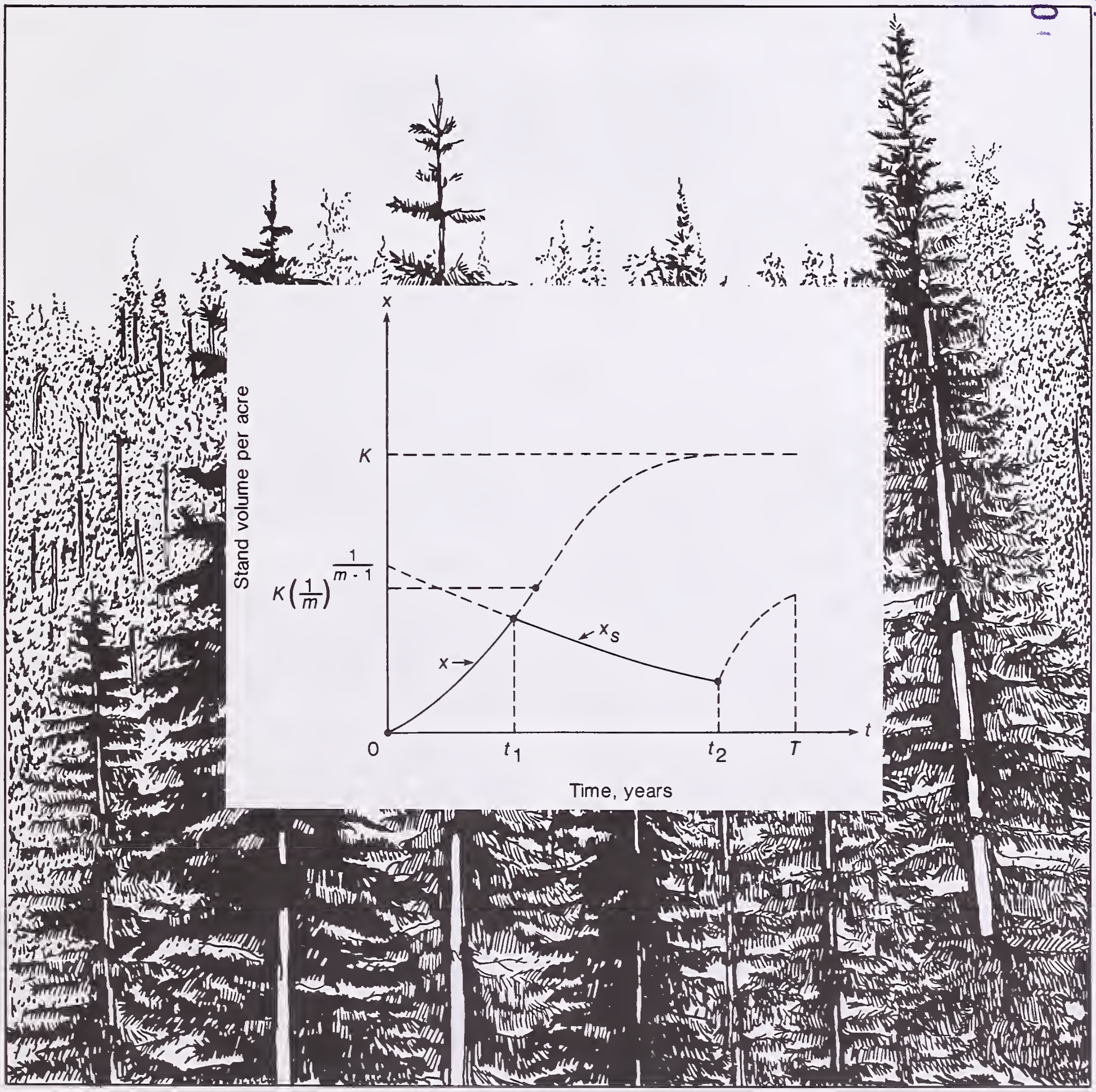

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Abstract

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Optimal control theory (OCT) is one method to determine when and how much to thin even-aged timber stands. Based on one common OCT application, the volume level of the residual stand follows what is called the singular path. The analysis shows how the singular path changes as stand growth and financial parameters change.

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Dennis M. Donnelly, David R. Betters, Matthew T. Turner, and Robert E. Gaines

Introduction

Optimal control theory (OCT) is an approach that has been used in a wide variety of time-dependent problems in areas such as manufacturing, aerospace, and economics. In the forestry context, OCT and other mathematical techniques allow the concurrent consideration of time-based stand growth concepts and dynamic economic concepts such as discount rates. OCT is a useful technique when forest managers see economic efficiency as a factor in making decisions about when and how much to cut from even-aged stands.

A classic treatment of even-aged timber stands includes one or more thinning cuts and later a final clearcut harvest, followed by regeneration. Many approaches are available to analyze how timber stands respond to such silvicultural treatments. One such approach is to model growth by a suitable mathematical function based on time. This allows introduction of economic considerations, especially the questions of when to thin, how much to thin, and when to make the final harvest (Betters et al. 1991).

When OCT was first applied to forest stand harvest modeling, one advantage was the explicit inclusion of nonlinearity in the stand growth models (Naslund 1969, Schreuder 1971, Anderson 1976). In addition, this work occurred when forest stand growth and its relation to yield of timber products was the main concern of many post-World War II forest managers. Growth modeling that typically uses equations and tables based on regression analysis became widespread (Myers and Godsey 1968, Dahms 1983).

In the 1980s, forest and stand management issues became much more diverse, and thus more complex, than previous concerns that emphasized timber production. Since OCT analysis of even simple forest stand harvesting problems requires advanced mathematics, the additional complexity of multipurpose stand management put OCT analysis beyond the interest of all but a handful of researchers. OCT, which can incorporate either linear or nonlinear relationships, provides an opportunity to try nonlinear, dynamic relationships in growth and yield modeling—albeit with the addition of mathematical complexity.

Stand volume growth is continuous and can be described by continuous mathematical functions. However, many of the management actions that could be applied to a forest stand are concentrated in time relative to the stand's life span. Thus, continuous OCT, which includes continuous control functions as discussed in this paper, is not always the best model if the goal is to simulate stand reaction to management at a highly detailed level. The state of the art in realistic, detailed simulation and optimization currently is based on discrete, single-tree growth simulators (Stage 1973, Wykoff et al. 1982) and corresponding discrete optimization methods (Haight and Monserud 1990a and 1990b, Monserud and Haight 1990).

The purpose here is to illustrate an alternative tool for looking at the policy interactions of financial economics and forest stand growth. Such policy examination is typically at an administrative scale where fine-tuned management details become blurred or may not be necessary for analysts looking at an overall picture. At this extensive scale, continuous analysis provides a useful overview.

This paper looks at the behavior of OCT solutions to forest stand financial optimization problems. The analysis uses a model that contains a large number of specific cases, depending on the value of the model's parameters. The analysis also shows how solution behavior changes as stand growth parameters change and as the financial parameters change.

Background

The OCT approach embodies a time-based decision criterion—here, for example, net present value—in an equation called the objective functional. Simultaneously, the dynamic behavior of the timber stand is characterized in a growth model, typically one or more differential or difference equations that specify the state of the system (the even-aged stand) at any given time. The solution of this problem, i.e., the optimal control(s), is determined in part by the Hamiltonian equation that is derived from the original problem equations. In the context of financial efficiency for management of even-aged stands, the solution is typically a time-based schedule of volumes to cut and, of course, a time-based schedule of volume to keep in the residual stand. This last concept in OCT parlance is the "trajectory" of the residual stand over time. The shape of this trajectory depends on parameters of the growth model and on economic parameters.

In the context of OCT as related to the scheduling of even-aged stand harvests, thinning is defined as a control variable. When this control variable enters linearly into the OCT objective functional and the ensuing Hamiltonian

equations, the problem solution is singular.[2] In such circumstances the optimal solution path for the residual stand volume may traverse what is termed the singular path during some time interval within the rotation.

To further motivate this problem, suppose a stand is to be thinned according to the solution of an optimal control model of stand management (see equation [4]). Stand volume level specified over time by the optimal control solution follows the solid line in figure 1. From time 0 to t_1, the stand volume increases without thinning following the line labeled x. In a continuous world, thinning commences at time t_1, and is conducted at an intensity such that stand volume follows the singular path x_s from t_1, to t_2. At time t_2, the stand is either harvested or allowed to grow until later harvest at time T. The assumptions of the model, the parameters of equations [1] - [3], and interest and value rates all affect the times t_1, t_2, and T, the shape of the singular path, and the pattern of thinning between t_1, and t_2.

To get into the details of the analysis of singular path behavior, the remainder of this paper deals first with the generalized growth model used. The optimal control problem is then introduced along with its resultant singular path equation. These two sections provide the background needed to investigate the shape of the singular path. Following this analysis, examples illustrate the general results.

The Class of Growth Models

Pienaar and Turnbull (1973) demonstrate a sigmoid-type growth model they call the Chapman-Richards generalization of von Bertalanffy's growth model. In our paper, it is called the Generalized Chapman-Richards Function (GCRF). The model in differential equation form is

$$\dot{x} = \frac{r}{m-1} t^{-b} \left\{ x \left[1 - \left(\frac{x}{K} \right)^{m-1} \right] \right\} \qquad [1]$$

where

x = the state variable (e.g., volume in cubic measure per unit of land area) and $\dot{x}$ is its time-dependent rate of change (i.e., growth);
K = maximum value of x approached asymptotically;
t = time, in years;
m = parameter controlling the point of inflection of the volume sigmoid curve;
r = growth rate parameter; and
b = parameter governing the attenuation of growth due to the passing of time, rather than due to the accumulation of biomass density (more later about this parameter).

This differential equation can be solved by separation of variables to obtain an expression for x,

$$x = KE(t)^{-\frac{1}{m-1}} \qquad [2]$$

where

$$E(t) = \left\{ 1 + \left[\left(\frac{K}{x(t_0)} \right)^{m-1} - 1 \right] e^{-\frac{r}{1-b}\left(t^{1-b} - t_0^{1-b}\right)} \right\}, \qquad [3]$$

t_0 = the initial time, usually $t = 0$; and
$x(t_0)$ = state variable magnitude at time = t_0.

Certain values of parameter m in equations [2] and [3] allow the GCRF to assume the form of well-known growth models (Pienaar and Turnbull 1973). For $m = 0$, equation [2] becomes the monomolecular growth equation (Richards 1959); for $m = 2/3$, the von Bertalanffy model; and for $m = 2$, the time-dependent logistic form. One other form is also important. Although $m = 1$ produces an indeterminate mathematical form in equations [1] and [2], as $m \to 1$ the resulting equation in the limit is in the form of a time-dependent Gompertz function.

Using the GCRF, we can replicate the Gompertz and logistic growth curves Turner (1988) used as the basis for his analysis of the shape of the singular path. In addition, we can analyze the shape of the singular path for an infinite number of possible growth functions when m is free to take on a value in the semiclosed interval $[0, M)$, where in theory M can be arbitrarily large. However, as we shall see, values between 0 and 4 are most useful for modeling even-aged stands.

Practical limits on the parameters r, m, and b of the GCRF must be established; to help set these limits, the effects of these parameters on shape of the growth curve are illustrated in figure 2. There, the parameters r, m, and b give the GCRF

[2] *This terminology derives from the second-order matrix (a one-element matrix in problems with a single state and control variable) that turns out to have a singular (i.e., equal to zero) determinant when the matrix is inverted. A strong analogy to this condition exists in ordinary differential calculus. Given a linear equation, $y = ax + b$, the first derivative is $y' = a$, and the second derivative is $y'' = 0$.*

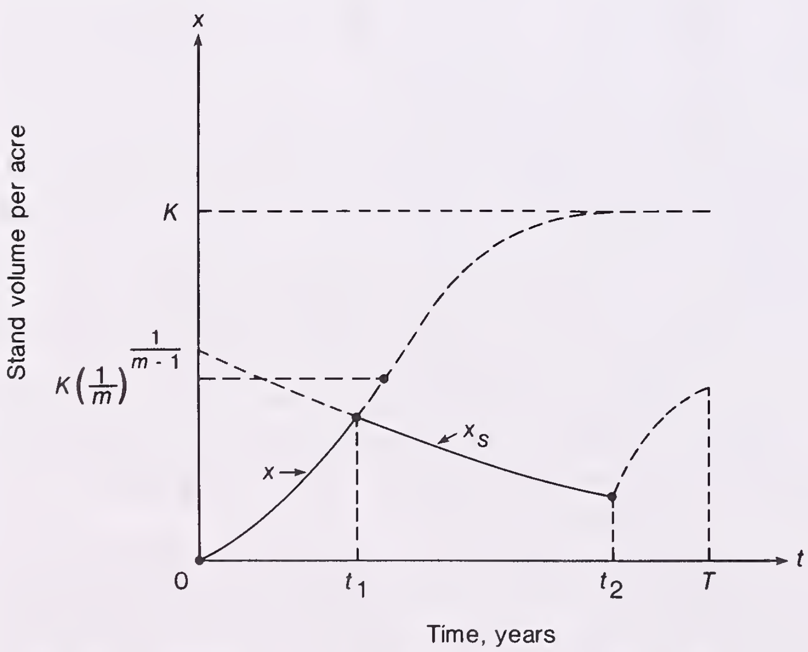

Figure 1.— Relation of the singular path, x_s, to stand growth, x.

curves various shapes that reasonably represent wide bounds on growth of even-aged forest stands. Thus, in this paper parameters r, m, and b have the following ranges:

$$0 < r \leq 1$$
$$0 \leq m \leq M\,(= 4)$$
$$0 \leq b < 0.9$$

Parameter r must be greater than zero to have positive growth in the GCRF model. As parameter r increases, growth and volume at a given stand age increase (fig. 2a). Arithmetically, r can be any real number greater than zero, but when r is greater than 1, growth depicted by the model is in excess of any range of real stand growth.

Inverse reasoning applies for values of parameter m greater than 4 (fig. 2b), since increasing m in the model slows simulated growth. Also, as noted earlier, the specific values of m that transform the GCRF into certain well known growth equation forms are 2 or less (Pienaar and Turnbull 1973).

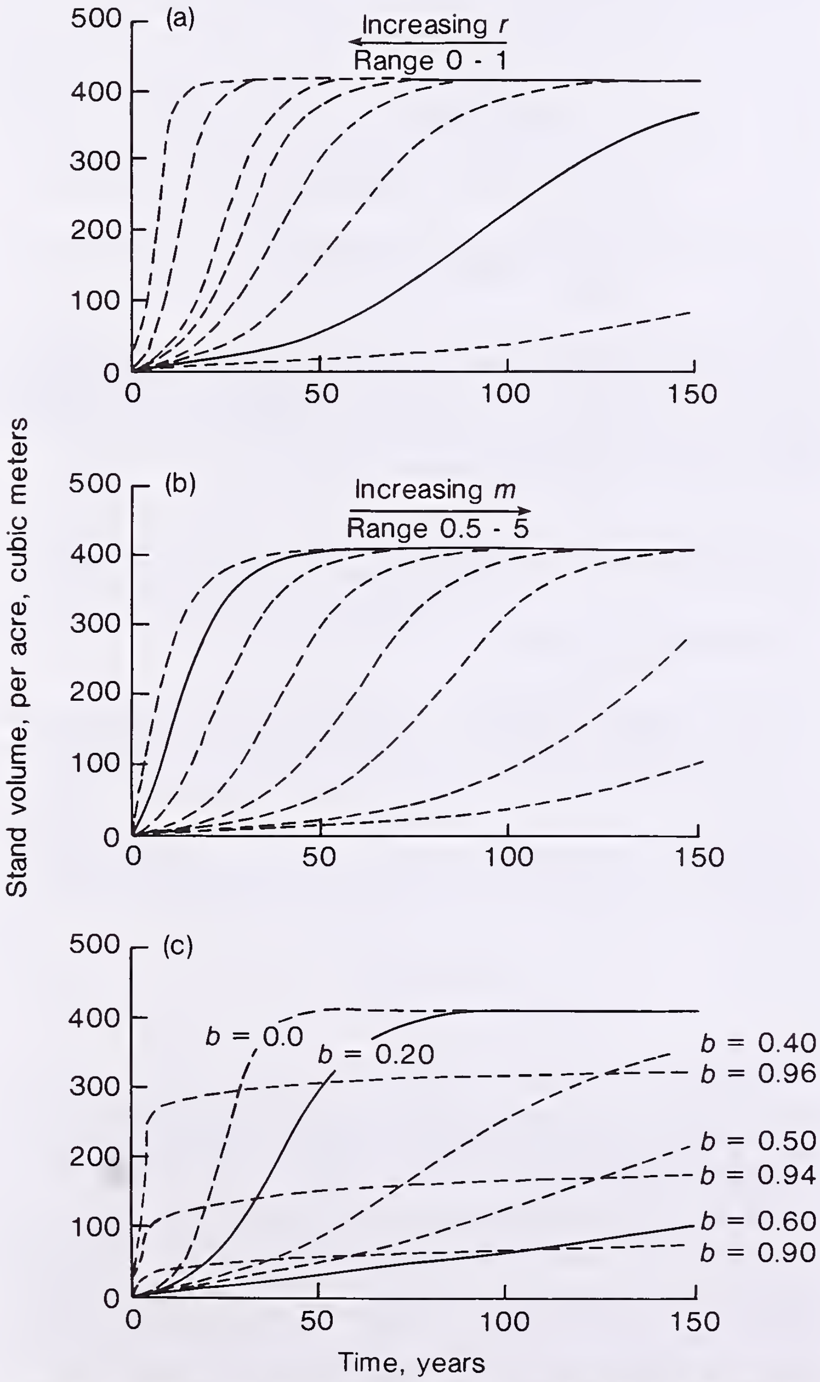

Figure 2.— Effect of GCRF parameters on volume model based on data and models in Kilkki and Vaisanen (1969): (a) effect of r on volume; (b) effect of m; and (c) effect of b.

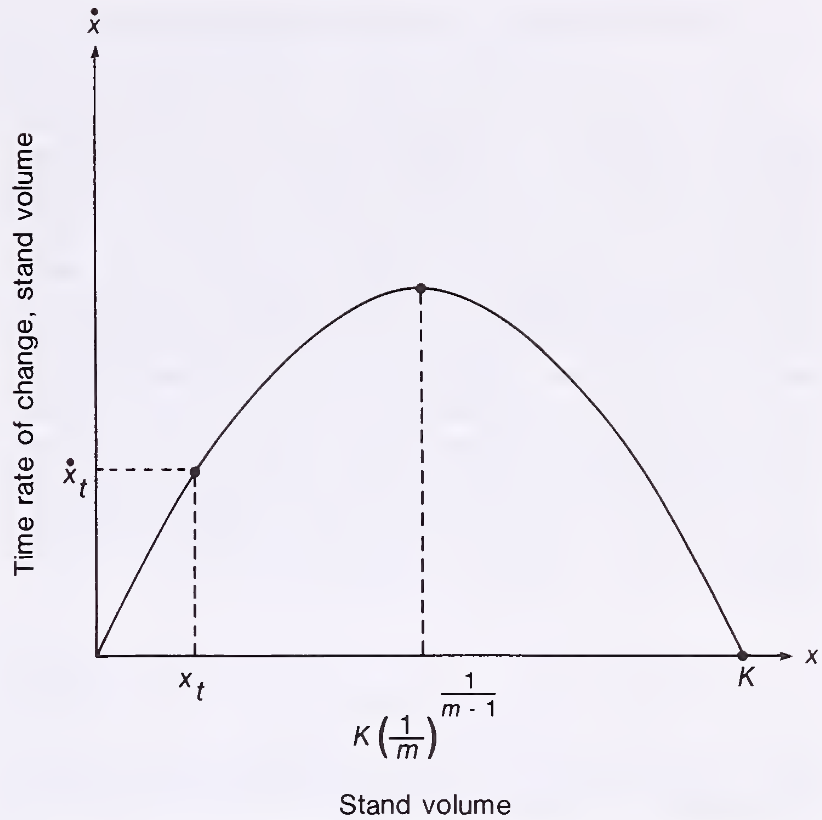

Figure 3.— First derivative of volume, $\dot{x}$, with respect to time, as related to stand volume, x.

Parameter b is incorporated into the GCRF model in the factor t^{-b} (Kilkki and Vaisanen 1969:6) because of the need for a function to attenuate growth under certain circumstances relating to the way stand management is modeled. Since stand growth, as modeled by density-dependent functions like the GCRF, is moderated by volume accumulation, models of unmanaged forest stands provide their own attenuation due to biomass increase. Consider equation [1], and for the moment let $b = 0$, i.e., the growth attenuation factor (GAF) t^{-b} does not affect the relationship. Without the GAF, growth in equation [1] depends entirely upon the stand volume, x. Behavior of x is shown generally in figure 3. The point of maximum growth with respect to x can be found by maximizing $\dot{x}$ with respect to x. However, assuming even-aged management as modeled by optimal control methods (Donnelly 1986; Donnelly, in prep.), it is mathematically possible within the model to maintain a stand, via thinning, at a level of volume such that growth would never slow. Figure 3 shows that if the modeled stand volume remains at $x = x_t$, due, say, to thinning, then $\dot{x}$ would always be at the same positive level, $\dot{x}_t$. However, it is more realistic that growth should diminish, not only due to accumulation of biomass and increased stand density (Davis and Johnson 1987:79) but also due to individual tree senescence (Spurr and Barnes 1980:393-5). This latter effect is modeled by the growth attenuation function (GAF) t^{-b}. Thus, adding the GAF t^{-b} to the model allows modeled growth to slow as time progresses regardless of the level of volume kept in the stand under management.

So for $0 \leq b < 0.9$, the parameter b, in concert with r and m, allows the GCRF to assume sigmoid shapes that fit a wide variety of tree growth data. However, we should note that for about $b > 0.9$, the GCRF model approaches and finally assumes certain limiting shapes not useful for tree growth modeling (fig. 2c).

The Stand Harvest Optimal Control Problem

This section provides a capsule statement of the stand harvest optimal control problem. For a more general treatment of this topic, see Donnelly (in prep.). Assume a rotation starts with planting or natural regeneration of a stand at time to t_0. The stand grows for a number of years until time t_1, when thinning starts. Thinning continues at a continuously variable rate $u(t)$ until time t_2. At this time the manager completely harvests the stand or waits several additional years for complete harvest. The complete harvest, when it occurs, is said to take place in year T. Then the cycle begins again. The problem is to find a control function $u(t)$ and control times t_1, t_2, and T that maximize net present value J per acre derived from the infinite series of even-aged stands. In equation form, the manager finds the maximum of

$$J= h(T)\left\{\int_{t_1}^{t_2} p(t)u(t)e^{-\delta t}\, dt + [q(T)x(t_2,T)e^{-\delta T} - C]\right\} \quad [4]$$

over an infinite number of sequential time intervals $[t_0, T]$, and subject to the state function,

$$\dot{x}= f(x,t) - u(t) \quad [5]$$

with the further specification that

$$x \geq 0;\ x(0) \approx 0;\ 0 < u(t) \leq u_{max}\ \forall u \in U. \quad [6]$$

Thus, $x(0)$ is arbitrarily close to, but not equal to, zero, and U is a class of admissible controls that are piecewise continuous in the interval $[t_1,T]$. The state equation (function) is typically a differential equation, for example, equation [1].

Additional notation for the optimal control problem is:

- J — a general symbol for the objective functional (i.e., equation [4]) in an OCT problem; $h(T)$ a factor based on the discount rate that accounts for perpetual use of land for forest stands in an infinite series of rotations; $h(T) = (1 - e^{\delta u})^{.2}$;
- $p(t)$ — net value of wood volume thinned at time t; $p(t) = Pe^{\rho t}$;
- $q(T)$ — net value of wood volume clearcut at time T; $q(T) = Qe^{\rho t}$;
- P,Q — base net values of wood per unit volume thinned and clearcut;
- ρ — rate of value increase of harvested roundwood due to premium accruing to larger material resulting from growth over time;
- $u(t)$ — rate of thinning in units of wood volume per unit of land area per year;
- δ — discount rate;
- t_1 — time at which thinning begins;
- t_2 — time at which thinning ends;
- T — time to harvest the residual stand, where $t_1 \leq t_2 \leq T$;
- $f(x,t)$ — notation for the functional equivalent of $\dot{x}$ in equation [1]; and
- C — fixed cost of regeneration in dollars per unit land area expended after each clearcut harvest.

The solution procedure uses the calculus of variations to form the so-called "first variation" of the objective functional adjoined by the state equation (Donnelly 1986; Donnelly, in prep). In a manner analogous to differential calculus, the first variation is set equal to zero in order to determine the necessary conditions to optimize the objective functional. Whether the optimal control function establishes a maximum, minimum, or stationary condition is determined by second-order conditions (which are not considered in this study) or by consideration of the OCT solution behavior.

For this particular problem, five necessary conditions are derived for an optimal objective functional J. These are,

$$\frac{\partial}{\partial u} H = \frac{\partial}{\partial u} \{u(t)[p(t)e^{-\delta t} - \lambda(t)] + \lambda(t)f(x,t)\} = 0 \quad [7]$$

where the Hamiltonian $H = p(t)u(t)\, e^{-\delta t} + \lambda(t)[f(x,t) - u(t)]$.

$$\dot{\lambda}(t) + \lambda(t)f_x(x,t) = 0 \quad [8]$$

$$\lambda(t_2) = q(T)e^{-\delta t} \quad [9]$$

$$p(t_2)u(t_2)e^{-\delta t_2} + q(T)\dot{x}_{t_2}(t_2,T)\mid_T e^{-\delta t} = \quad [10]$$

$$q(T)\, \dot{x}_t(t_2, T) \mid_T + \dot{q}(T)\, x(t_2, T)$$
$$= \delta h(T)\left\{\int_{t_1}^{t_2} p(t)u(t)e^{-\delta t}\, dt + [q(T)x(t_2, T) - C]\right\} \quad [11]$$

Each of the necessary conditions above has natural economic interpretations. The Appendix outlines how the terms in the necessary conditions reflect economic thought relative to biological growth.

Solutions

The Optimal Trajectory

The optimal trajectory is based on the first necessary condition, equation [7]. In order to maximize the Hamiltonian with respect to the control, $u(t)$, the first partial derivative of H with respect to u is set equal to zero, as in equation [7]. Solution of this procedure results in the expression,

$$\lambda(t) = p(t)\, e^{-\delta t} \quad [12]$$

However, when this condition holds, the Hamiltonian is no longer dependent on $u(t)$. In this situation, $u(t)$ and the optimal stand volume trajectory follow the singular path, yet to be determined.

Since the time interval when equation [12] holds is not yet determined, the possibility exists that over some other time interval, $\lambda(t)$ is not equal to $p(t)e^{-\delta t}$. If $\lambda(t) < p(t)e^{-\delta t}$, then H is maximized by allowing $u(t)$ to be its maximum rate, i.e., thinning the stand at the maximum rate allowed by the constraints. Conversely, if $\lambda(t) > p(t)^{e-\delta t}$, then H is maximized by setting $u(t) = 0$, i.e., not thinning the stand at all.

These conditions are summarized as

$$u(t) = \begin{cases} 0 & \text{for } \lambda(t) > p(t)e^{-\delta t} \\ u_s(t) & \text{for } \lambda(t) = p(t)e^{-\delta t} \\ u_{max} & \text{for } \lambda(t) < p(t)e^{-\delta t} \end{cases} \qquad [13]$$

It is the equality in equation [13] that must be satisfied on the singular path. Thus, it is the solution of the adjoint differential equation that provides the value of $\lambda(t)$, which in turn determines the value of $u(t)$ according to equation [13]. However, at this stage of the analysis it is not known if or when the trajectory follows the singular path. For now, presume the singular path is a major feature of the solution.

The adjoint differential equation (equation [8]) must be satisfied throughout the time interval $[0,T]$ including any subintervals in which the singular path might be part of the solution. When the solution follows the singular path, from equation [13], $\lambda_s(t)=p(t)e^{-\delta t}$, where the subscript s denotes a singular path relationship. Since $\lambda_s(t)$ is a solution to the adjoint differential equation, the equations for $\lambda_s(t)$ (just given) and its first derivative $\dot{\lambda}_s(t)$ can be substituted into equation [8] to obtain an expression for the response of the state variable $x_s(t)$ on the singular path.

To do this, take the first derivative of $\lambda_s(t)$ from equation [12] with respect to t, getting

$$\dot{\lambda}_s(t) = [e^{-\delta t}\dot{p}(t) - \delta p(t)\, e^{-\delta t} \qquad [14]$$

Substitute $\lambda_s(t)$ and $\dot{\lambda}_s(t)$ into equation [8], and cancel eligible terms to get

$$\rho - \delta + f_x(x,t) = 0 \qquad [15]$$

Find $f_x(x,t)$ from equation [1] by partially differentiating its right-hand side with respect to x and substitute the result into equation [15], giving

$$\rho - \delta + \left\{ \frac{r}{m-1}\, t^{-b} \left[1 - m \left(\frac{x}{K} \right)^{m-1} \right] \right\} = 0 \qquad [16]$$

Solve this expression for x, which becomes x_s the volume trajectory on the singular path. The resulting equation is

$$x_s = K \left[\frac{1}{m} - \frac{(\rho-\delta)\,(m-1)}{rm}\, t^b \right]^{\frac{1}{m-1}} \qquad [17]$$

This expression depends on parameters of the GCRF (i.e., K, r, b, and m) and is a function of time. It also depends on the rate ρ that stand value increases due to tree growth, and the rate of interest used as a discount factor δ.

Shape of the Singular Path

The next step is to look at what shapes the singular path can take depending on the magnitudes of its parameters. Two additional relationships needed for this analysis are the first and second derivatives of the singular path equation [17] with respect to time. These are

$$\dot{x}_s = -KF_1^{-\frac{m-2}{m-1}}\, F_2 \qquad [18]$$

and

$$\ddot{x}_s = \underbrace{\left[KF_1^{-\frac{m-2}{m-1}} \right]}_{\text{Term 1}} \underbrace{\left[\frac{1-b}{t}\, F_2 + (2-m)\, F_1^{-1} F_2^2 \right]}_{\text{Term 2}} \qquad [19]$$

The labels "Term 1" and "Term 2" are used later. In simplified form,

$$\ddot{x}_s = -\,x_s \left[\frac{(1-b)}{t} + (2-m)\frac{F_2}{F_1} \right] \qquad [20]$$

where

$$F_1 = \frac{1}{m} - \frac{(m-1)(\delta-\rho)}{mr}\, t^b \qquad [21]$$

and

$$F_2 = \frac{\delta-\rho}{mr}\, bt^{b-1} \qquad [22]$$

Several factors in equations [18], [19] and [20] are the same, as shown by the notation F_1 and F_2. In addition, the term in brackets in equation [17] is factor F_1 (as defined in equation [21]. This similarity is useful as discussed next.

Examination of the singular path's first and second derivatives (equations [18] and [19]) over the selected ranges of b, r, and m provides information about the behavior of the singular path under different combinations of the parameters from the stand growth model, equation [1]. In equation [19], the terms in brackets are labeled Term 1 and Term 2, respectively. The signs of Terms 1 and 2 determine the sign of the singular path second derivative and, consequently, the shape of the singular path with respect to the origin.

Intuitively, at a specific point in time, if the signs of Terms 1 and 2 are either both positive or both negative, the second derivative is positive (singular path convex); if the sign of only one of the terms is negative, then the second derivative is negative (singular path concave).

However, the situation is more complex. In the first panel of figure 4a note that the singular path is positive in the first quadrant of time-volume space and then, after the singular path crosses the time axis, it is negative in the fourth quadrant. The range of interest for harvest scheduling applications is the positive values of the first quadrant.

In equation [19] we've noted that factor F_1 within brackets in Term 1 is the same as the expression within brackets in the singular path equation (equation [17]). The overall similarity between the singular path equation (equation [17]) and Term 1 of its second derivative (equation [19]) means that Term 1 must always be either positive or zero for the solution to be in the positive first quadrant of the (t, x) solution space.

To find the time the singular path crosses the t axis, set equation [17] = 0 and solve for t_s. The solution is

$$t_s = \left[\frac{r}{(\delta-\rho)(m-1)} \right]^{\frac{1}{b}} \qquad [23]$$

where t_s is the time the singular path crosses the time axis, i.e., $x_s = 0$ for $t = t_s$ and $x_s > 0$ for $t < t_s$. So, Term 1 of equation [19] is positive or equal to zero for $t \leq t_s$ because of its similarity with the term in brackets in equation [17], the positive singular path.

Thus, the sign of Term 2 determines the sign of the singular path second derivative and, consequently, the singular path shape. By setting Term 2 of equation [19] = 0, the values of time t can be determined for which Term 2 is less than, equal to, or greater than zero. Let $t = t_i$ be the time when Term 2 equals zero and solve the equation formed by setting Term 2 = 0.

$$\frac{1-b}{t_i} F_2(t_i) - (2-m) F_1^{-1}(t_i) F_2^2(t_i) = 0 \qquad [24]$$

and

$$t_i = \left[\frac{(1-b)r}{(\delta-\rho)(m-b-1)} \right]^{\frac{1}{b}} \qquad [25]$$

Because of the complexity of the Term 2 = 0 expression, it may not be monotonic and so would have more than one possible value for t_i. However, in the time interval [0, T] of interest in this problem, the risk can be assessed empirically by looking at the graphs of the singular path and its first and second derivative.

The following relationships exist in the first quadrant of time-volume space between time t and t_i, the sign of x_s, and the shape of the singular path.

When time t is —	Term 2 is —	the second derivative x_s is —	and the singular path is —
$< t_i$	> 0	> 0	convex to the origin
$= t_i$	$= 0$	$= 0$	a stationary point
$> t_i$	< 0	< 0	concave to the origin

Refer to figure 4. Time t_s is the instant when the singular path crosses the time axis, i.e., when $x_s = 0$. Time t_i denotes an inflection point in the singular path when its shape changes from convex to concave, or vice versa (fig. 4). It is possible that both, only one, or neither of these special times (t_s and t_i) exist, as we shall see next.

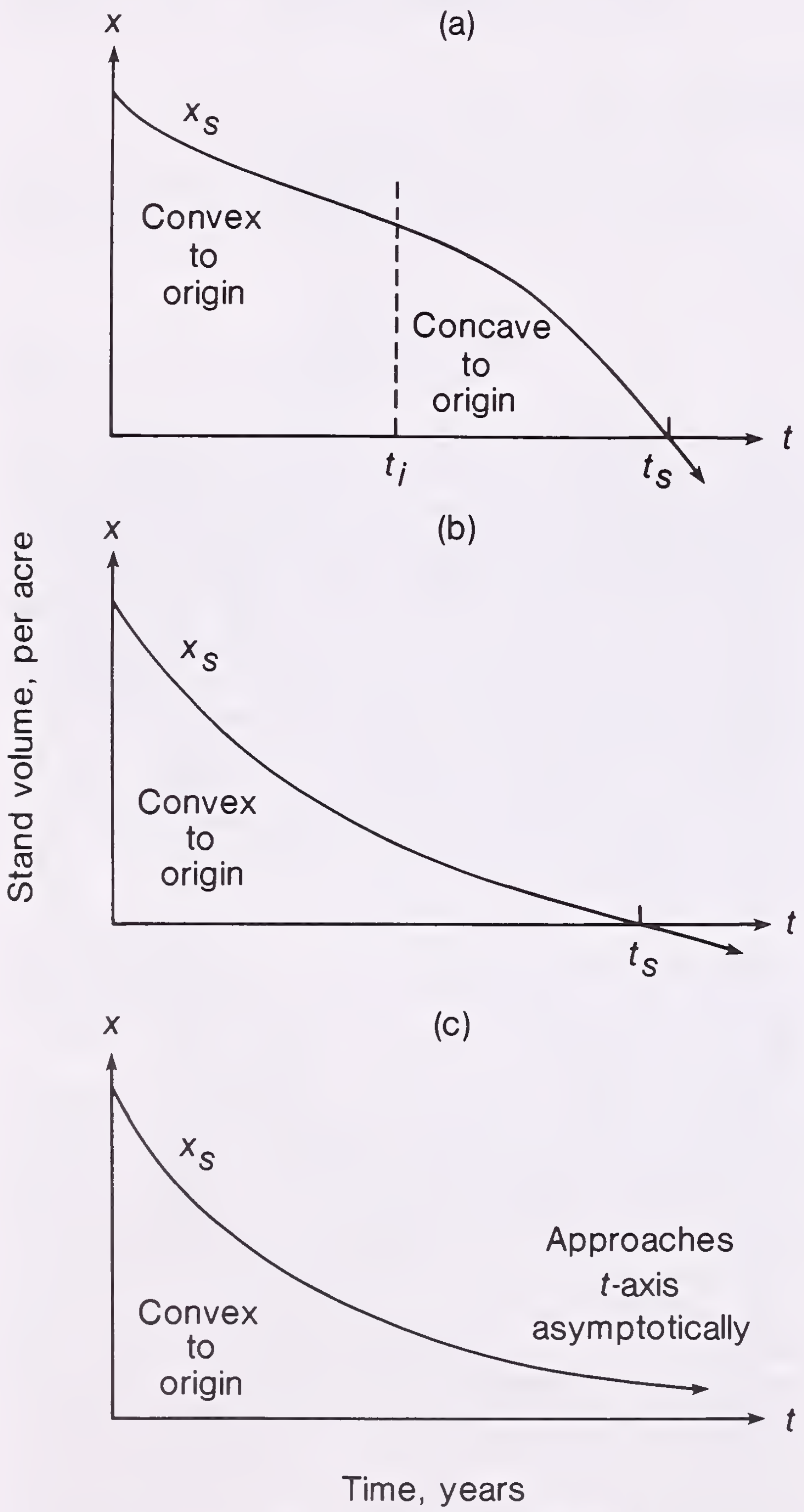

Figure 4.— Possible configurations of the singular path with respect to time (*t*) and volume (*x*) space.

Existence of t_s and t_i

Having expressions for the key times that affect the behavior of the singular path second derivative, we can look at how the parameters r, b, and m control t_s and t_i. Parameters r, b, and m in equations [23] and [25] come originally from the GCRF. Their magnitude is determined by the statistical fit of the GCRF model to the data that describe stand growth. In addition to the general ranges for b, r, and m noted earlier, we also temporarily assume that $(\delta-\rho) > 0$; that $r > 0$ always; and that b can take values from 0 to approximately 0.9. Given the results shown is figure 2, these assumptions seem reasonable and do not unduly restrict the scope of the analysis.

Because of the structure of equations [23] and [25], certain values for parameters b and m determine the existence of the two key times, t_s and t_i. But first note that because parameter b is restricted to the range $0 < b < 0.9$, the exponent $1/b$ in equations [23] and [25] takes values in the range $[1.11 < (1/b) < \infty]$. This relationship is important be-

cause in the following discussion the term with the exponent $1/b$ in equations [23] and [25] may be positive or negative. The sign of this term is crucial in determining the existence of inflection points.

To see how this works, consider equation [23]. Assume that $(\delta - \rho) > 0$. The case where $(\delta - \rho) < 0$ is considered later. In equation [23], the operand term is infinite for $m = 1$ and negative for $m < 1$. In either case, exponentiation by $1/b$ *is* not defined for values of b generally in its range. Equation [23] can be evaluated arithmetically only for $m > 1$. However, there are exceptions to this general case. If the value of b is such that $1/b$ is any positive integer, then exponentiation is mathematically possible no matter what the sign of the operand. But even if exponentiation is technically possible when the operand is negative, the result is not meaningful in the context of determining an inflection point. To sum up, if $m \leq 1$ in equation [23], then that equation and t_s generally do not exist because the singular path does not cross the time axis.

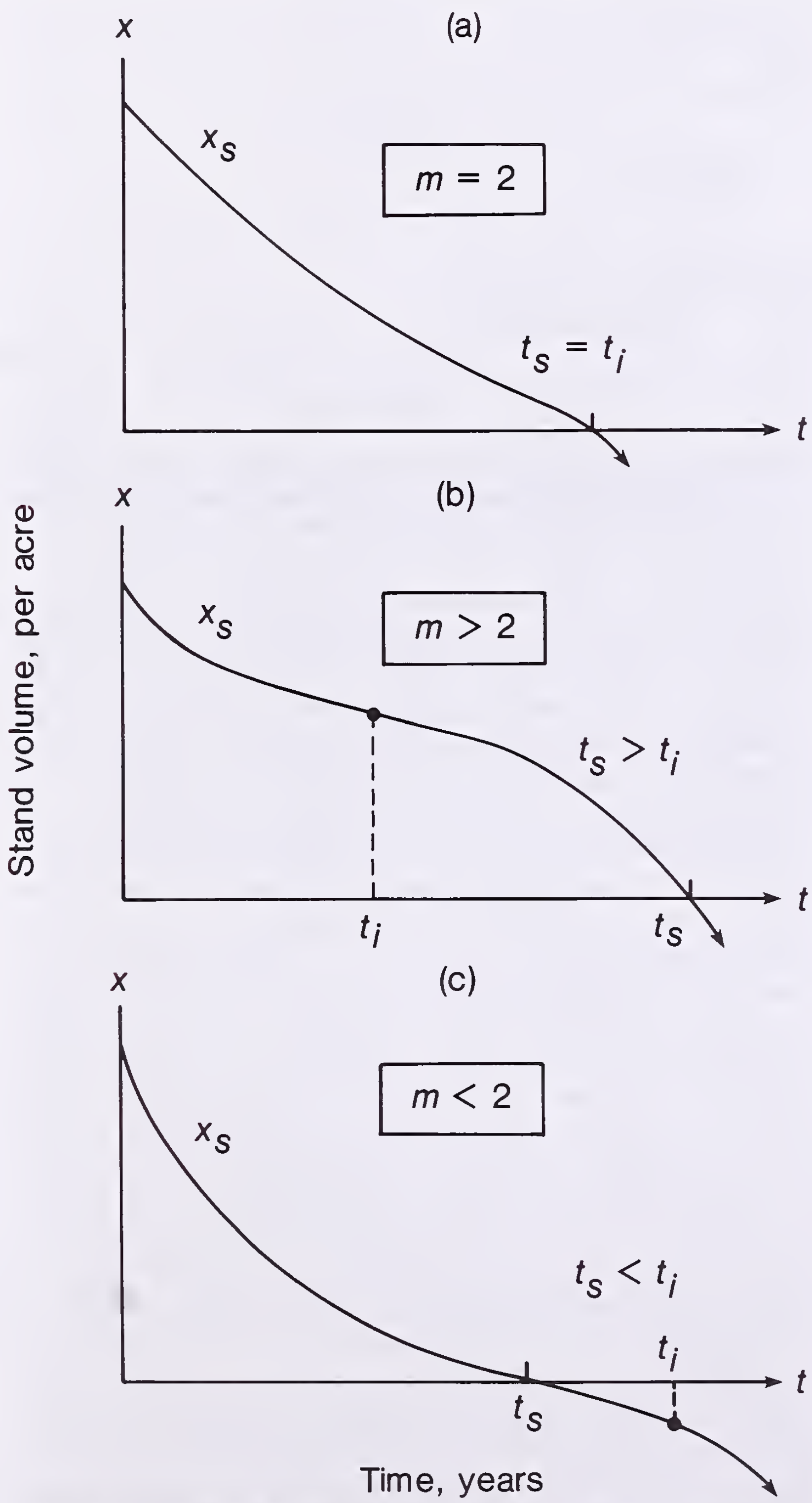

Figure 5.— Possible relationships between t_s and t_i for values of the GCRF parameter *m*.

Equation [25] is more important in determining existence of an inflection point. Since we've restricted the value of b to the range $0 < b < 0.9$, the term $(1-b) > 0$. However, factor $(m-b-1)$ in the denominator of equation [25] is negative if $m < (b + 1)$. Thus, if equation [25] is to exist and indicate the time t_i of an inflection point, the inequality $m > (b + 1)$ must hold.

Considering equations [23] and [25] together, note that for $m = 2$, $t_s = t_i$. Further, if $m > 2$, $t_s > t_i$, and if $m < 2$, then $t_s < t_i$ (fig. 5). This means that parameter m must be greater than 2 if there is to be any chance of an inflection point and a concave singular path occurring during the time scale of a rotation. In other words, the inflection point must occur in time before the singular path crosses the time axis (if the singular path does so at all; see figures 4 and 5).

The relationships between parameters b, r, and m are summarized in figure 6. For values of b and m in region 1 (R1), $m > (b + 1)$, so both a zero crossing, t_s, and a point of inflection, t_i, can exist (see figures 4a and 5b). For values in region 2 (R2), only a zero crossing exists, not a point of inflection (fig. 4b). In region 3 (R3), neither a zero crossing nor a point of inflection exists (fig. 4c). Thus, in order to determine if the singular path is concave to the origin, the analysis reduces to asking what combinations of parameters b and m would 1) place the singular path solution in region 1 on figure 6; 2) make the second derivative of the singular path negative; and 3) allow the time of zero crossing, t_s, *to* be less than some user-defined upper time limit T for tree growth age within a rotation.

Examples

Turner's (1988) results are based on two classic sigmoid curves that are special cases of the GCRF. The logistic curve comes from setting $m = 2$ in the GCRF, and the Gompertz curve results from assessing the limit of the GCRF as m approaches 1. Turner's data for the logistic curve are for lodgepole pine *(Pinus contorta* Dougl. ex Loud.) in the

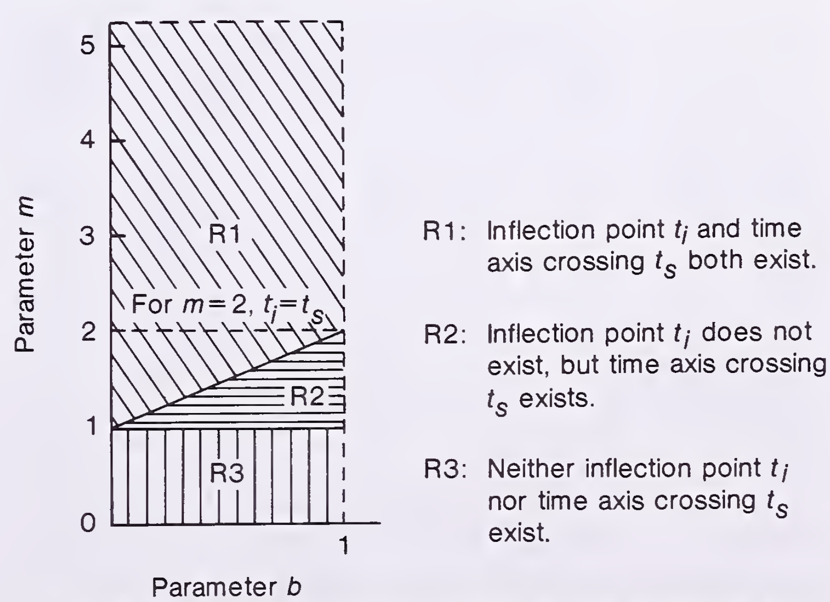

Figure 6.— Parameters *b, r,* and *m* determine the existence and practicality of an inflection point in the singular path.

interior Pacific Northwest (Dahms 1964). The following examples assume $\delta > \rho$, and their results are summarized in table 1.

Using nonlinear regression, Dahms' data were fit to the GCRF. There is no presupposition about the form being either logistic ($m = 2$) or Gompertz ($m \rightarrow 1$). Results of this nonlinear regression fit are shown in figure 7a. The best fit regression has parameter $m = 1.11$ and parameter $b = 0.5$. The values of these two parameters fall into region R2 of figure 6. Earlier derivation indicates that if $1 < m \leq b + 1$, the singular path (fig. 7b) crosses the time axis but has no inflection point. Numerical results show that the singular path does indeed intersect the time axis at 4996 years! This particular singular path is always convex to the origin in terms of any stand rotation.

Using the GCRF from this example as a point of departure, the analysis can be varied in several ways to further illustrate how convexity or concavity can occur in the singular path associated with the class of OCT models considered in this paper. Three cases with selected values for parameters b and m are used to compare behavior of the singular path (fig. 8).

Case one.—Let $m = 2.2$ and $b = 0.5$. These parameter values fall into the region R1 of figure 6. According to equations [23] and [25], respectively, the singular path intersects the time axis at $t_s = 156$ years and inflection occurs is at $t_i = 115$ years. The singular path is concave to the origin from 115 years to 156 years. Figure 8a shows the volume curve and the singular path that would result if a statistically fitted volume curve displays the parameters assumed for this case.

Case two.—Assume the fitted growth curve now has values for b and m such that $1 < m \leq b + 1$. The parameter values for b and m in this instance fall into region R2 of figure 6. The descending singular path is tangential to the time axis at $t_s = 124$ years as shown in figure 8b. Beyond 124 years, the singular path rises but this part of the curve is not relevant in terms of stand harvest. The singular path is convex over the entire relevant time span of 0 to 124 years.

Table 1.—List of example parameters for the GCRF along with corresponding states of the singular path.

	GCRF stand model parameter			Times of interest		Parameter region in figure 6
	m	b	r	t_i	t_s	
Dahm's (1964) lodgepole pine example (GCRF)	1.110	0.500	0.155	none	4996	2
Parameter change examples (based on LP example)						
Case one	2.200	0.500	0.300	115	156	1
Case two	1.500	0.600	0.180	none	124	2
Case three	0.800	0.600	0.180	none	none	3

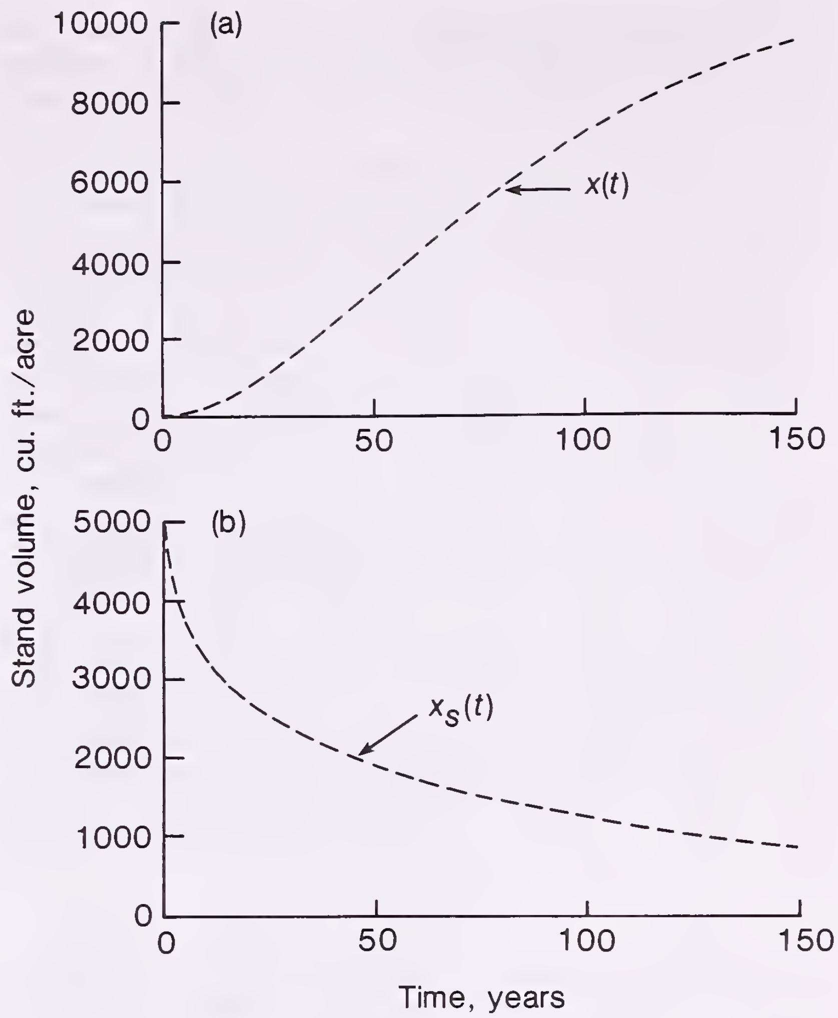

Figure 7.— Volume and singular path curves based on yield tables for lodgepole pine from the interior Pacific Northwest (Dahms 1964). GCRF parameters: $m = 1.110$; $b = 0.5$; $r = 0.155$.

Case three.—Assume the fitted growth curve has parameters b and m such that $m < 1$ and $0 < b < 1$. The parameter values for this case fall into region R3 of figure 6. In this case, the singular path curve does not intersect the time axis at any finite time, nor does it have an inflection point. The singular path is thus convex to the origin at all times and neither time t_s nor time t_i exist (fig. 8c).

After considering these examples, it should be noted that there is one other way the singular path can be concave to the origin. An assumption in this paper is that $(\delta - \rho) > 0$ but this assumption need not be as we noted earlier. If ρ is greater than δ, the implication is that the rate of value increase exceeds the rate of discount. Referring back to equation [4], the integrand is $p(t)u(t)e^{-\delta t}$. In terms of units, this term is dollars per acre per year, discounted from the future time to the present. The factor $p(t) = Pe^{\rho t}$ is comprised of a base price P (\$/volume) that increases each year by a factor $e^{\rho t}$. Stand and tree characteristics that allow lower fixed costs per unit harvested presumably make the stand more valuable as time passes. The coefficient ρ reflects this premium value attached to a forest stand as it increases in volume and individual tree size. In the case where $(\delta - \rho) < 0$, the singular path increases over time rather than decreases. Since the singular path volume at $t = 0$ is greater than the stand volume in this analysis, it is possible that the volume curve does not intersect the singular path within any number of years included in the biological life span of the stand.

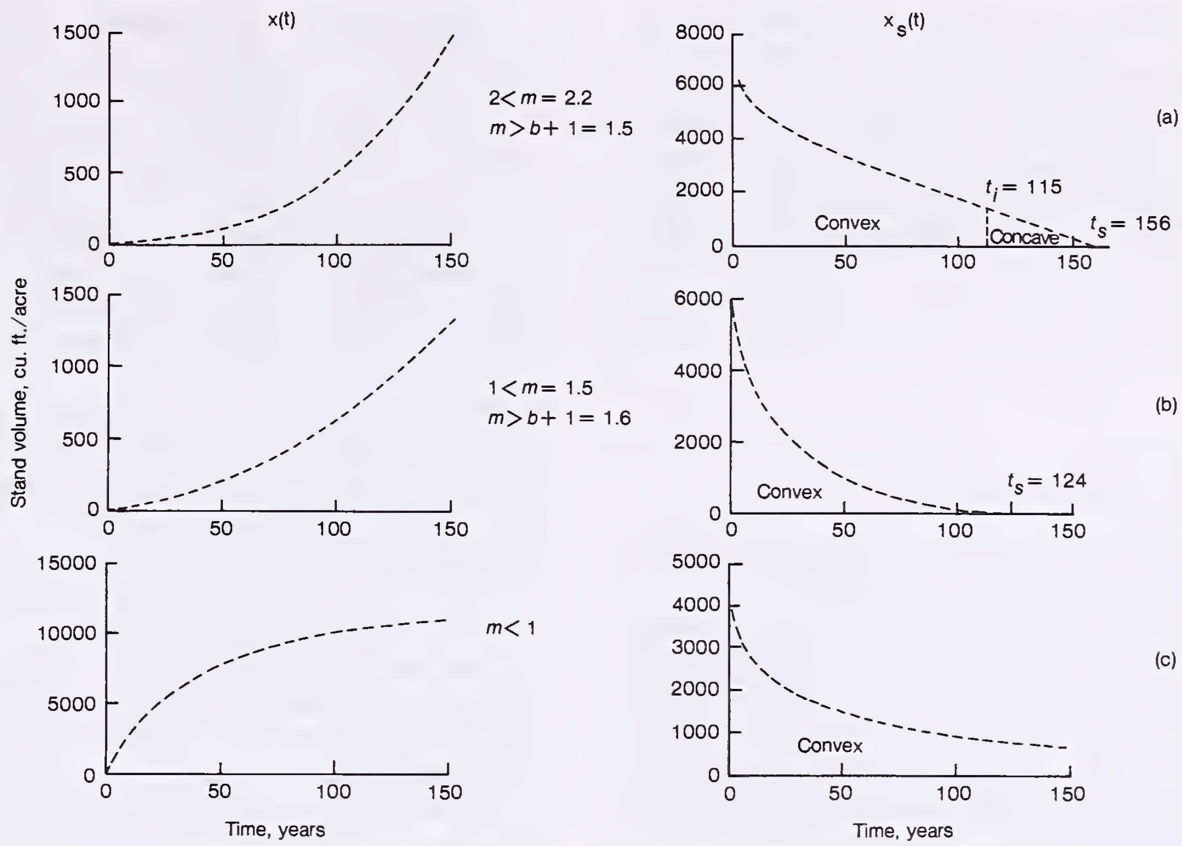

Figure 8.— Volume and singular path curves with GCRF parameters set to illustrate conditions of concavity and convexity in the singular path.

In figure 9, which again depicts the results for Dahms' lodgepole pine example, the lowest singular path is shown by the line for δ-ρ>> 0. This line is the same as in the lower panel of figure 7. The other singular path lines in figure 8 show what happens as δ-ρ progressively decreases (δ-ρ> 0), goes to zero (δ-ρ = 0), and then becomes negative as $\rho > \delta$ (δ-ρ < 0) and (δ-ρ << 0). In the last two instances the singular path is concave to the origin and intersects the natural volume curve at times much greater than the instances when δ - ρ is greater than or equal to zero.

Economically, the increased growing period indicated when $\rho > \delta$ is a result of the fact that present net value is occurring on the stump faster than when $\delta > \rho$. Thus, letting the stand grow longer makes economic sense within the context of the assumptions for this study.

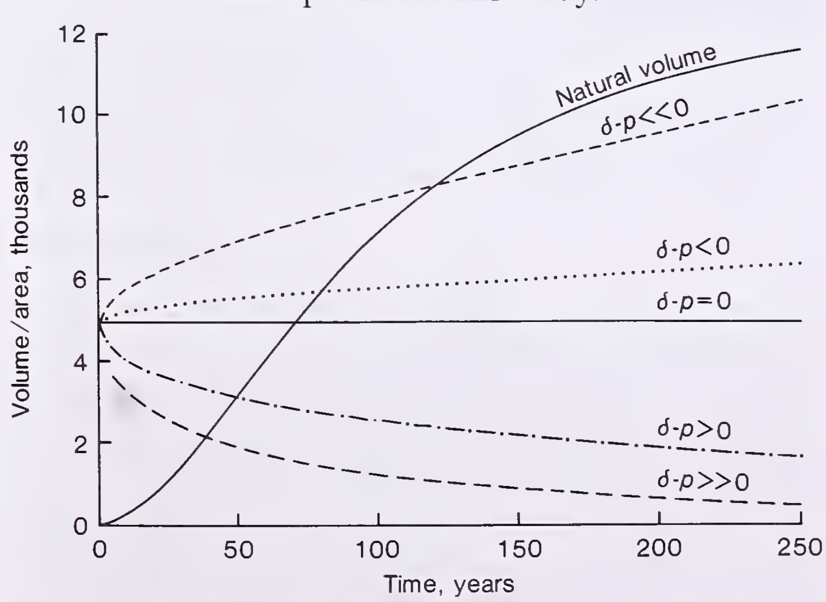

Figure 9.— Volume curve and singular path curves with economic parameters δ and ρ set to show effect of δ less than, equal to, or greater than ρ.

Conclusions

Caution is essential when dealing with models of any process, and especially with models of stand growth. It is entirely too easy to confuse the modeled process with the real process. The relevance of this concern lies in recognizing that an optimal control format for harvesting strategy, including concepts such as the singular path, is contingent upon the validity of the underlying biological and economic models. Thus, in this paper, the shape and form of the singular path is valid assuming that the GCRF validly portrays stand growth over the parameter (b, m, r) ranges considered. Shape and form also are influenced by assuming that discounted value (related to δ) and tree growth value (related to ρ) are relevant economic influences in the decision about when and how much to harvest.

Magnitudes of the biological parameters (b, m, r) in the GCRF model depend entirely on its fit to the stand data. In turn, these same parameters determine the shape of the singular path, i.e., the volume magnitude that the optimally-controlled timber stand should have through time until the final harvest. Keeping timber stand volume on or near the optimal trajectory (singular path) requires thinning. Specifying precisely how much and when to thin is not within the scope of this paper (see Donnelly in prep.). But the amount to thin at any given point in time is determined in part by the difference between what stand volume is and what it optimally should be. Hence, singular path shape is important.

For many configurations of the GCRF, especially including well-known functional forms such as the Gompertz and logistic equations, the associated singular path is entirely convex to the origin. Larger values of m in the GCRF imply the possibility of a volume inflection point, i.e.,

maximum growth, later in stand life. To whatever extent even-aged forest stands attain their maximum growth later in their life spans, the possibility opens for a concave singular path. However, if most even-aged stands achieve maximum growth early, then m is likely 2 or less, precluding the possibility of a concave singular path. So, while a concave singular path is possible within the GCRF model, it is also unlikely.

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Appendix

Economic Interpretation

The Objective Functional

The objective functional (equation [4]) gives present net value (PNV) per unit of land area (such as hectare or acre) generated by economic use of a timber stand with inputs of time, labor (thinning or cutting), and capital (growing stock).

In equation [4], the integrand is the flow of PNV achieved by thinning from time t_1 to time t_2, and is measured in terms of value per unit land area per unit of time (such as dollars per acre per year).

The terminal value term in equation [4], $[q(T)x(t_2,T)e^{-\rho t} - C]$, would be called the salvage value in many OCT economics applications. In the context of forest harvest, it is the value of the residual stand after thinning and, most likely, after a period of post-thinning growth. This term represents the PNV of the volume clearcut at the end of the rotation—net of the cost C per unit land area needed to regenerate the stand—and its units of measurement are value per land area.

Thus, the total present net value of managing the stand is an additive combination of PNV from thinning and final clearcut. These two sources of PNV imply there is the potential for a trade-off of PNV generated from thinning with that generated from the final clearcut. How this trade-off is resolved is determined by factors such as the discount rate, stand growth, and relative prices for thinned and clearcut wood volume.

The State Equation

The state equation, $x = f(x,t) - u(t)$, is basically a production relationship. Its output is volume change per unit time (or in its integrated form, total volume at a specific time). Its inputs are stock of physical capital, labor in the form of potential thinning, and time.

When the stand is thinned, trees considered unproductive are removed. Variable inputs are applied to the most productive units of capital, i.e., the remaining fastest-growing, best quality trees in the stand. The change in growth between a thinned stand and an unthinned stand then represents the marginal contribution made by the increments of input factors that went into thinning.

The Hamiltonian

The Hamiltonian function, a key part of the solution to the stand harvest scheduling problem, has units of measurement of dollars per hectare per year. It is a measure of the time flow of PNV per unit area based on inputs to the production function (the state equation). It is the instantaneous rate of total contribution of PNV to the objective functional *J*.

To examine this contribution from a slightly different perspective, consider the PNV contribution to *J* during a small time interval from *t* to *t+dt*. The Hamiltonian is then written,

$$Hdt = p(t)\,u(t)\,e^{-\delta t}\,dt + \lambda(t)\,[f(x,t) - u(t)]\,dt$$

The first term on the right-hand side is the direct PNV contribution to *J* during time from *t* to *t+dt* if the system has capital assets *x* (volume) and control level $u(t)$ is applied. The units of measurement for this term are dollars per unit land area. The second term is the PNV of change in capital assets (growing stock) during time *t* to *t+dt*. It is an indirect contribution and is valued by the presence of the adjoint function which serves as a surrogate "net value."

The Adjoint Function

The adjoint function, as represented by the adjoint variable λ, is a surrogate "shadow price" for a unit of capital employed in the production process. As the production process (stand growth) proceeds, the surrogate capital value is implicitly compared to value available by converting capital into goods, i.e., wood products. This process is depicted by the adjoint differential equation, equation [8].

The differential form of equation [8] is $d\lambda = -\lambda(t)f_{x'}(x,t)dt$. This equation gives marginal change in net value $(d\lambda)$ of a unit of capital, (such as cubic volume of growing stock) during a time interval $[t, t+dt]$ while traversing the optimal path and applying the optimal control $u(t)$.

Terminal Time Equations

The two terminal time equations (equations [10] and [11]) taken together define the time (t_2) to stop thinning and the time (T) to end the rotation by clearcutting. The first terminal time equation (equation [10]) shows marginal changes at t_2 and the second expression (equation [11]) shows marginal changes at *T*.

Consider equation [10]:

$$p(t_2)u(t_2)e^{-\delta t} + q(T)\dot{x}_{t_2}(t_2,T)\,|_{T}\,e^{-\delta t} = 0$$

The first term is the marginal change in PNV from thinning cuts as a result of a marginal change in the time to stop thinning (t_2). The second term is the marginal change in PNV value from the final clearcut as a result of a marginal change in the time to stop thinning (t_2). For example, a differential increase in t_2, i.e., t_2+dt_2, will result in an increment of value added to *J* from the slightly prolonged thinning period. But there may be a decrement in the value of *J* due to the slight decrease in volume at *T*. How these two changes relate to each other in value depends on parameters of the particular problem.

Now look at equation [11]:

$$q(T)\,\dot{x}_T(t_2, T)\,|_T + q(T)\,\dot{x}(t_2, T)$$
$$= \frac{\delta}{1-e^{-\delta t}}\left\{\int_{t_1}^{t_2} p(t)\,u(t)\,e^{-\delta t}\,dt + [q(T)\,x(t_2, T) - C]\right\}$$

The left-hand side (LHS) of this equation is the total marginal change in value due to a slight shift in terminal

time, for example, from T to $T+dT$. This differential shift in terminal time has two effects. The first term of the LHS is the marginal change in value due to a time-based change in volume at T. The second term of the LHS is the marginal change due to a time-based change in value at T. The right-hand side (RHS) of this equation is the PNV opportunity cost of delay of one time unit in all future rotations. The integral term is the value from thinning in each future rotation, and the term $[q(T)x(t_2,T)-C]$ is the value from the final harvest in each future rotation. The sum of these two terms is multiplied by the discount rate δ to give an opportunity cost for each future rotation. This quantity is then capitalized over all future rotations by multiplying by the factor $h(T)=(1-e^{-\delta t})^{-1}$.

Donnelly, Dennis M.; Betters, David R.; Turner, Matthew T.; Gaines, Robert E. 1992. Thinning Even-Aged Forest Stands: Behavior of Singular Path Solutions in Optimal Control Analyses. Res. Pap. RM-307. Fort Collins, CO: U.S. Department of Agriculture, Forest Service, Rocky Mountain Forest and Range Experiment Station. 12 p.

Optimal control theory (OCT) is one method to determine when and how much to thin even-aged timber stands. Based on one common OCT application, the volume level of the residual stand follows what is called the singular path.

Keywords: mathematical model, optimal control, analyses, forest growth, economics

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